Online Appendix to "Collusion through Coordination of Announcements"

Joseph E. Harrington, Jr.* and Lixin Ye†

29 October 2018

Section 6: Collusion

Profit Expression

To show that

$$\begin{aligned} & B(h,t_1;h,t_2) - B(\phi(t_1),t_1;\phi(t_2),t_2) \\ = & \int_{\underline{v}}^{\overline{v}} \int_{R_{t_1t_2}(v)}^{R_{HH}(v)} \int_{\underline{c}_{t_1}}^{R_{t_1t_2}(v)} \left(c_2 - R_{t_1t_2}(v)\right) dF_{t_1}(c_1)\, dF_{t_2}(c_2)\, dG(v) \\ & + \int_{\underline{v}}^{\overline{v}} \int_{R_{t_1t_2}(v)}^{R_{HH}(v)} \int_{R_{t_1t_2}(v)}^{c_2} \left(c_2 - c_1\right) dF_{t_1}(c_1)\, dF_{t_2}(c_2)\, dG(v) \\ & + \int_{\underline{v}}^{\overline{v}} \int_{R_{HH}(v)}^{\bar{c}_{t_2}} \int_{\underline{c}_{t_1}}^{R_{t_1t_2}(v)} \left(R_{HH}(v) - R_{t_1t_2}(v)\right) dF_{t_1}(c_1)\, dF_{t_2}(c_2)\, dG(v) \\ & \int_{\underline{v}}^{\overline{v}} \int_{R_{HH}(v)}^{\bar{c}_{t_2}} \int_{R_{t_1t_2}(v)}^{R_{HH}(v)} \left(R_{HH}(v) - c_1\right) dF_{t_1}(c_1)\, dF_{t_2}(c_2)\, dG(v), \end{aligned}$$

consider

*Department of Business Economics & Public Policy, The Wharton School, University of Pennsylvania, Philadelphia, PA 19104, harrij@wharton.upenn.edu

†Department of Economics, Ohio State University, Columbus, OH 43210, ye.45@osu.edu

$$
\begin{aligned}
& B(h,t_1;h,t_2) - B(t_1,t_1;t_2,t_2) \\
= & \int_{\underline{v}}^{\overline{v}} \int_{R_{t_1t_2}(v)}^{R_{HH}(v)} \int_{\underline{c}_{t_1}}^{c_2} (c_2 - c_1)\, dF_{t_1}(c_1)\, dF_{t_2}(c_2)\, dG(v) \\
& + \int_{\underline{v}}^{\overline{v}} \int_{R_{HH}(v)}^{\bar{c}_{t_2}} \int_{\underline{c}_{t_1}}^{R_{LL}(v)} (R_{HH}(v) - R_{t_1t_2}(v))\, dF_{t_1}(c_1)\, dF_{t_2}(c_2)\, dG(v) \\
& + \int_{\underline{v}}^{\overline{v}} \int_{R_{HH}(v)}^{\bar{c}_{t_2}} \int_{R_{LL}(v)}^{R_{t_1t_2}(v)} (R_{HH}(v) - R_{t_1t_2}(v))\, dF_{t_1}(c_1)\, dF_{t_2}(c_2)\, dG(v) \\
& + \int_{\underline{v}}^{\overline{v}} \int_{R_{HH}(v)}^{\bar{c}_{t_2}} \int_{R_{t_1t_2}(v)}^{R_{HH}(v)} (R_{HH}(v) - c_1)\, dF_{t_1}(c_1)\, dF_{t_2}(c_2)\, dG(v) \\
& - \int_{\underline{v}}^{\overline{v}} \int_{R_{t_1t_2}(v)}^{R_{HH}(v)} \int_{\underline{c}_{t_1}}^{R_{LL}(v)} (R_{t_1t_2}(v) - c_1)\, dF_{t_1}(c_1)\, dF_{t_2}(c_2)\, dG(v) \\
& - \int_{\underline{v}}^{\overline{v}} \int_{R_{t_1t_2}(v)}^{R_{HH}(v)} \int_{R_{LL}(v)}^{R_{t_1t_2}(v)} (R_{t_1t_2}(v) - c_1)\, dF_{t_1}(c_1)\, dF_{t_2}(c_2)\, dG(v)
\end{aligned}
$$

and the following sequence of steps

$$
\begin{aligned}
& B(h,t_1;h,t_2) - B(t_1,t_1;t_2,t_2) \\
= & \int_{\underline{v}}^{\overline{v}} \int_{R_{t_1t_2}(v)}^{R_{HH}(v)} \int_{\underline{c}_{t_1}}^{R_{LL}(v)} (c_2 - c_1)\, dF_{t_1}(c_1)\, dF_{t_2}(c_2)\, dG(v) \\
& + \int_{\underline{v}}^{\overline{v}} \int_{R_{t_1t_2}(v)}^{R_{HH}(v)} \int_{R_{LL}(v)}^{c_2} (c_2 - c_1)\, dF_{t_1}(c_1)\, dF_{t_2}(c_2)\, dG(v) \\
& + \int_{\underline{v}}^{\overline{v}} \int_{R_{HH}(v)}^{\bar{c}_{t_2}} \int_{\underline{c}_{t_1}}^{R_{LL}(v)} (R_{HH}(v) - R_{t_1t_2}(v))\, dF_{t_1}(c_1)\, dF_{t_2}(c_2)\, dG(v) \\
& + \int_{\underline{v}}^{\overline{v}} \int_{R_{HH}(v)}^{\bar{c}_{t_2}} \int_{R_{LL}(v)}^{R_{t_1t_2}(v)} (R_{HH}(v) - R_{t_1t_2}(v))\, dF_{t_1}(c_1)\, dF_{t_2}(c_2)\, dG(v) \\
& + \int_{\underline{v}}^{\overline{v}} \int_{R_{HH}(v)}^{\bar{c}_{t_2}} \int_{R_{t_1t_2}(v)}^{R_{HH}(v)} (R_{HH}(v) - c_1)\, dF_{t_1}(c_1)\, dF_{t_2}(c_2)\, dG(v) \\
& - \int_{\underline{v}}^{\overline{v}} \int_{R_{t_1t_2}(v)}^{R_{HH}(v)} \int_{\underline{c}_{t_1}}^{R_{LL}(v)} (R_{t_1t_2}(v) - c_1)\, dF_{t_1}(c_1)\, dF_{t_2}(c_2)\, dG(v) \\
& - \int_{\underline{v}}^{\overline{v}} \int_{R_{t_1t_2}(v)}^{R_{HH}(v)} \int_{R_{LL}(v)}^{R_{t_1t_2}(v)} (R_{t_1t_2}(v) - c_1)\, dF_{t_1}(c_1)\, dF_{t_2}(c_2)\, dG(v)
\end{aligned}
$$

$$
\begin{aligned}
& B(h,t_1;h,t_2) - B(t_1,t_1;t_2,t_2) \\
= \; & \int_{\underline{v}}^{\overline{v}} \int_{R_{t_1t_2}(v)}^{R_{HH}(v)} \int_{\underline{c}_{t_1}}^{R_{LL}(v)} \left(c_2 - R_{t_1t_2}(v)\right) dF_{t_1}(c_1)\, dF_{t_2}(c_2)\, dG(v) \\
& + \int_{\underline{v}}^{\overline{v}} \int_{R_{t_1t_2}(v)}^{R_{HH}(v)} \int_{R_{LL}(v)}^{c_2} (c_2 - c_1)\, dF_{t_1}(c_1)\, dF_{t_2}(c_2)\, dG(v) \\
& + \int_{\underline{v}}^{\overline{v}} \int_{R_{HH}(v)}^{\bar{c}_{t_2}} \int_{\underline{c}_{t_1}}^{R_{LL}(v)} \left(R_{HH}(v) - R_{t_1t_2}(v)\right) dF_{t_1}(c_1)\, dF_{t_2}(c_2)\, dG(v) \\
& + \int_{\underline{v}}^{\overline{v}} \int_{R_{HH}(v)}^{\bar{c}_{t_2}} \int_{R_{LL}(v)}^{R_{t_1t_2}(v)} \left(R_{HH}(v) - R_{t_1t_2}(v)\right) dF_{t_1}(c_1)\, dF_{t_2}(c_2)\, dG(v) \\
& + \int_{\underline{v}}^{\overline{v}} \int_{R_{HH}(v)}^{\bar{c}_{t_2}} \int_{R_{t_1t_2}(v)}^{R_{HH}(v)} \left(R_{HH}(v) - c_1\right) dF_{t_1}(c_1)\, dF_{t_2}(c_2)\, dG(v) \\
& - \int_{\underline{v}}^{\overline{v}} \int_{R_{t_1t_2}(v)}^{R_{HH}(v)} \int_{R_{LL}(v)}^{R_{t_1t_2}(v)} \left(R_{t_1t_2}(v) - c_1\right) dF_{t_1}(c_1)\, dF_{t_2}(c_2)\, dG(v)
\end{aligned}
$$

$$
\begin{aligned}
& B(h,t_1;h,t_2) - B(t_1,t_1;t_2,t_2) \\
= \; & \int_{\underline{v}}^{\overline{v}} \int_{R_{t_1t_2}(v)}^{R_{HH}(v)} \int_{\underline{c}_{t_1}}^{R_{LL}(v)} \left(c_2 - R_{t_1t_2}(v)\right) dF_{t_1}(c_1)\, dF_{t_2}(c_2)\, dG(v) \\
& + \int_{\underline{v}}^{\overline{v}} \int_{R_{t_1t_2}(v)}^{R_{HH}(v)} \int_{R_{LL}(v)}^{R_{t_1t_2}(v)} (c_2 - c_1)\, dF_{t_1}(c_1)\, dF_{t_2}(c_2)\, dG(v) \\
& + \int_{\underline{v}}^{\overline{v}} \int_{R_{t_1t_2}(v)}^{R_{HH}(v)} \int_{R_{t_1t_2}(v)}^{c_2} (c_2 - c_1)\, dF_{t_1}(c_1)\, dF_{t_2}(c_2)\, dG(v) \\
& + \int_{\underline{v}}^{\overline{v}} \int_{R_{HH}(v)}^{\bar{c}_{t_2}} \int_{\underline{c}_{t_1}}^{R_{LL}(v)} \left(R_{HH}(v) - R_{t_1t_2}(v)\right) dF_{t_1}(c_1)\, dF_{t_2}(c_2)\, dG(v) \\
& + \int_{\underline{v}}^{\overline{v}} \int_{R_{HH}(v)}^{\bar{c}_{t_2}} \int_{R_{LL}(v)}^{R_{t_1t_2}(v)} \left(R_{HH}(v) - R_{t_1t_2}(v)\right) dF_{t_1}(c_1)\, dF_{t_2}(c_2)\, dG(v) \\
& + \int_{\underline{v}}^{\overline{v}} \int_{R_{HH}(v)}^{\bar{c}_{t_2}} \int_{R_{t_1t_2}(v)}^{R_{HH}(v)} \left(R_{HH}(v) - c_1\right) dF_{t_1}(c_1)\, dF_{t_2}(c_2)\, dG(v) \\
& - \int_{\underline{v}}^{\overline{v}} \int_{R_{t_1t_2}(v)}^{R_{HH}(v)} \int_{R_{LL}(v)}^{R_{t_1t_2}(v)} \left(R_{t_1t_2}(v) - c_1\right) dF_{t_1}(c_1)\, dF_{t_2}(c_2)\, dG(v)
\end{aligned}
$$

$$
\begin{aligned}
& B(h,t_1;h,t_2) - B(t_1,t_1;t_2,t_2) \\
= \; & \int_{\underline{v}}^{\overline{v}} \int_{R_{t_1t_2}(v)}^{R_{HH}(v)} \int_{\underline{c}_{t_1}}^{R_{LL}(v)} \left(c_2 - R_{t_1t_2}\left(v\right)\right) dF_{t_1}\left(c_1\right) dF_{t_2}\left(c_2\right) dG\left(v\right) \\
& + \int_{\underline{v}}^{\overline{v}} \int_{R_{t_1t_2}(v)}^{R_{HH}(v)} \int_{R_{LL}(v)}^{R_{t_1t_2}(v)} \left(c_2 - R_{t_1t_2}\left(v\right)\right) dF_{t_1}\left(c_1\right) dF_{t_2}\left(c_2\right) dG\left(v\right) \\
& + \int_{\underline{v}}^{\overline{v}} \int_{R_{t_1t_2}(v)}^{R_{HH}(v)} \int_{R_{t_1t_2}(v)}^{c_2} \left(c_2 - c_1\right) dF_{t_1}\left(c_1\right) dF_{t_2}\left(c_2\right) dG\left(v\right) \\
& + \int_{\underline{v}}^{\overline{v}} \int_{R_{HH}(v)}^{\overline{c}_{t_2}} \int_{\underline{c}_{t_1}}^{R_{LL}(v)} \left(R_{HH}\left(v\right) - R_{t_1t_2}\left(v\right)\right) dF_{t_1}\left(c_1\right) dF_{t_2}\left(c_2\right) dG\left(v\right) \\
& + \int_{\underline{v}}^{\overline{v}} \int_{R_{HH}(v)}^{\overline{c}_{t_2}} \int_{R_{LL}(v)}^{R_{t_1t_2}(v)} \left(R_{HH}\left(v\right) - R_{t_1t_2}\left(v\right)\right) dF_{t_1}\left(c_1\right) dF_{t_2}\left(c_2\right) dG\left(v\right) \\
& + \int_{\underline{v}}^{\overline{v}} \int_{R_{HH}(v)}^{\overline{c}_{t_2}} \int_{R_{t_1t_2}(v)}^{R_{HH}(v)} \left(R_{HH}\left(v\right) - c_1\right) dF_{t_1}\left(c_1\right) dF_{t_2}\left(c_2\right) dG\left(v\right)
\end{aligned}
$$

$$
\begin{aligned}
& B(h,t_1;h,t_2) - B(t_1,t_1;t_2,t_2) \\
= \; & \int_{\underline{v}}^{\overline{v}} \int_{R_{t_1t_2}(v)}^{R_{HH}(v)} \int_{\underline{c}_{t_1}}^{R_{t_1t_2}(v)} \left(c_2 - R_{t_1t_2}\left(v\right)\right) dF_{t_1}\left(c_1\right) dF_{t_2}\left(c_2\right) dG\left(v\right) \\
& + \int_{\underline{v}}^{\overline{v}} \int_{R_{t_1t_2}(v)}^{R_{HH}(v)} \int_{R_{t_1t_2}(v)}^{c_2} \left(c_2 - c_1\right) dF_{t_1}\left(c_1\right) dF_{t_2}\left(c_2\right) dG\left(v\right) \\
& + \int_{\underline{v}}^{\overline{v}} \int_{R_{HH}(v)}^{\overline{c}_{t_2}} \int_{\underline{c}_{t_1}}^{R_{t_1t_2}(v)} \left(R_{HH}\left(v\right) - R_{t_1t_2}\left(v\right)\right) dF_{t_1}\left(c_1\right) dF_{t_2}\left(c_2\right) dG\left(v\right) \\
& \int_{\underline{v}}^{\overline{v}} \int_{R_{HH}(v)}^{\overline{c}_{t_2}} \int_{R_{t_1t_2}(v)}^{R_{HH}(v)} \left(R_{HH}\left(v\right) - c_1\right) dF_{t_1}\left(c_1\right) dF_{t_2}\left(c_2\right) dG\left(v\right).
\end{aligned}
$$

Welfare Expression

Expected total welfare under competition is

$$q\left[{}^2 b\int_{\underline{c}_L}^{\bar{c}_L}\int_{R_L^{-1}(c)}^{\bar{v}}(v-c)\,dG(v)\,dF_L(c)+\right. \tag{1}$$

$$\left.(1-b)\int_{\underline{c}_L}^{\bar{c}_L}\int_{\underline{c}_L}^{\bar{c}_L}\int_{R_{LL}^{-1}(\min\{c_1,c_2\})}^{\bar{v}}(v-\min\{c_1,c_2\})\,dG(v)\,dF_L(c_1)\,dF_L(c_2)\right]$$

$$2q(1-q)\left[b\int_{\underline{c}_L}^{\bar{c}_L}\int_{R_L^{-1}(c)}^{\bar{v}}(v-c)\,dG(v)\,dF_L(c)+\right.$$

$$\left.(1-b)\int_{\underline{c}_H}^{\bar{c}_H}\int_{\underline{c}_L}^{\bar{c}_L}\int_{R_{LH}^{-1}(\min\{c_1,c_2\})}^{\bar{v}}(v-\min\{c_1,c_2\})\,dG(v)\,dF_L(c_1)\,dF_H(c_2)\right]$$

$$+(1-q)^2\left[b\int_{\underline{c}_H}^{\bar{c}_H}\int_{R_H^{-1}(c)}^{\bar{v}}(v-c)\,dG(v)\,dF_H c+\right.$$

$$\left.(1-b)\int_{\underline{c}_H}^{\bar{c}_H}\int_{\underline{c}_H}^{\bar{c}_H}\int_{R_{HH}^{-1}(\min\{c_1,c_2\})}^{\bar{v}}(v-\min\{c_1,c_2\})\,dG(v)\,dF_H(c_1)\,dF_H(c_2)\right]$$

Expected total welfare under collusion is

$$q^2\left[b\int_{\underline{c}_L}^{\bar{c}_L}\int_{R_H^{-1}(c)}^{\bar{v}}(v-c)\,dG(v)\,dF_L(c)+\right. \tag{2}$$

$$\left.(1-b)\int_{\underline{c}_L}^{\bar{c}_L}\int_{\underline{c}_L}^{\bar{c}_L}\int_{R_{HH}^{-1}(\min\{c_1,c_2\})}^{\bar{v}}(v-\min\{c_1,c_2\})\,dG(v)\,dF_L(c_1)\,dF_L(c_2)\right]$$

$$+2q(1-q)\left[b\left(\frac{1}{2}\right)\int_{\underline{c}_L}^{\bar{c}_L}\int_{R_H^{-1}(c)}^{\bar{v}}(v-c)\,dG(v)\,dF_L(c)+\right.$$

$$\left.\left(\frac{1}{2}\right)\int_{\underline{c}_H}^{\bar{c}_H}\int_{R_H^{-1}(c)}^{\bar{v}}(v-c)\,dG(v)\,dF_H(c)\right]$$

$$+2q(1-q)(1-b)\left[\int_{\underline{c}_H}^{\bar{c}_H}\int_{\underline{c}_L}^{\bar{c}_L}\int_{R_{HH}^{-1}(\min\{c_1,c_2\})}^{\bar{v}}(v-\min\{c_1,c_2\})\,dG(v)\,dF_L(c_1)\,dF_H(c_2)\right]$$

$$+(1-q)^2\left[b\int_{\underline{c}_H}^{\bar{c}_H}\int_{R_H^{-1}(c)}^{\bar{v}}(v-c)\,dG(v)\,dF_H(c)+\right.$$

$$\left.(1-b)\int_{\underline{c}_H}^{\bar{c}_H}\int_{\underline{c}_H}^{\bar{c}_H}\int_{R_{HH}^{-1}(\min\{c_1,c_2\})}^{\bar{v}}(v-\min\{c_1,c_2\})\,dG(v)\,dF_H(c_1)\,dF_H(c_2)\right]$$

Subtracting (1) from (2) gives us the change in expected total welfare due to collusion. Re-arranging and simplifying this expression yields

$$\begin{aligned}
\Delta(q) \quad \equiv \quad & q^2 \left[\int_{\underline{c}_L}^{\min\{\overline{c}_L, R_H(\overline{v})\}} \int_{R_H^{-1}(c)}^{\overline{v}} (v-c)\, dG(v)\, dF_L(c) - \right. \\
& \left. \int_{\underline{c}_L}^{\min\{\overline{c}_L, R_L(\overline{v})\}} \int_{R_L^{-1}(c)}^{\overline{v}} (v-c)\, dG(v)\, dF_L(c) \right] \\
& +2q(1-q) \left[\left(\frac{1}{2}\right) \int_{\underline{c}_L}^{\min\{\overline{c}_L, R_H(\overline{v})\}} \int_{R_H^{-1}(c)}^{\overline{v}} (v-c)\, dG(v)\, dF_L(c) + \right. \\
& \left(\frac{1}{2}\right) \int_{\underline{c}_H}^{\min\{\overline{c}_H, R_H(\overline{v})\}} \int_{R_H^{-1}(c)}^{\overline{v}} (v-c)\, dG(v)\, dF_H(c) - \\
& \left. \int_{\underline{c}_L}^{\min\{\overline{c}_L, R_L(\overline{v})\}} \int_{R_L^{-1}(c)}^{\overline{v}} (v-c)\, dG(v)\, dF_L(c) \right]
\end{aligned} \tag{3}$$

If $b = 1$ then

$$\begin{aligned}
\Delta(q) \quad = \quad & q^2 \left[\int_{\underline{c}_L}^{\overline{c}_L} \int_{R_H^{-1}(c)}^{R_L^{-1}(c)} (v-c)\, dG(v)\, dF_L(c) \right] \\
& +2q(1-q) \left[\left(\frac{1}{2}\right) \int_{\underline{c}_L}^{\overline{c}_L} \int_{R_H^{-1}(c)}^{R_L^{-1}(c)} (v-c)\, dG(v)\, dF_L(c) + \right. \\
& \left(\frac{1}{2}\right) \int_{\underline{c}_H}^{\overline{c}_H} \int_{R_H^{-1}(c)}^{\overline{v}} (v-c)\, dG(v)\, dF_H(c) - \\
& \left. \left(\frac{1}{2}\right) \int_{\underline{c}_L}^{\overline{c}_L} \int_{R_L^{-1}(c)}^{\overline{v}} (v-c)\, dG(v)\, dF_L(c) \right].
\end{aligned} \tag{4}$$

Next assume the distributions in Section 6. In that case,

$$R_L(v) = \left(\frac{\alpha}{\alpha+1}\right) v \text{ and } R_L^{-1}(c) = \left(\frac{\alpha+1}{\alpha}\right) c$$

$$R_H(v) = \left(\frac{\beta}{\beta+1}\right) v \text{ and } R_H^{-1}(c) = \left(\frac{\beta+1}{\beta}\right) c.$$

(4) becomes

$$\begin{aligned}\Delta(q) \;=\; & q^2\left[\int_0^{\frac{\beta}{\beta+1}}\left(\int_{\left(\frac{\beta+1}{\beta}\right)c}^{1}(v-c)\,dv\right)\alpha c^{\alpha-1}dc-\int_0^{\frac{\alpha}{\alpha+1}}\left(\int_{\left(\frac{\alpha+1}{\alpha}\right)c}^{1}(v-c)\,dv\right)\alpha c^{\alpha-1}dc\right]\\ & +2q(1-q)\left[\left(\frac{1}{2}\right)\int_0^{\frac{\beta}{\beta+1}}\left(\int_{\left(\frac{\beta+1}{\beta}\right)c}^{1}(v-c)\,dv\right)\alpha c^{\alpha-1}dc+\right.\\ & \left(\frac{1}{2}\right)\left(\int_0^{\frac{\beta}{\beta+1}}\int_{\left(\frac{\beta+1}{\beta}\right)c}^{1}(v-c)\,dv\right)\beta c^{\beta-1}dc-\\ & \left.\int_0^{\frac{\alpha}{\alpha+1}}\left(\int_{\left(\frac{\alpha+1}{\alpha}\right)c}^{1}(v-c)\,dv\right)\alpha c^{\alpha-1}dc\right].\end{aligned}$$

This is the expression that is evaluated in Figure 2.

Section 7: Collusion for a Class of Parametric Distributions

As all buyers approach only one seller, optimal reserve prices (depending on the seller's revealed type) are

$$R_L(v)=\arg\max_R(v-R)F_L(R)=\arg\max_R(v-R)R^{\alpha}\Rightarrow R_L(v)=\left(\frac{\alpha}{\alpha+1}\right)v.$$

$$R_H(v)=\arg\max_R(v-R)F_H(R)=\arg\max_R(v-R)R^{\beta}\Rightarrow R_H(v)=\left(\frac{\beta}{\beta+1}\right)v.$$

It can be shown that a buyer's expected utility is $\frac{\alpha^{\alpha}v^{\alpha+1}}{(\alpha+1)^{\alpha+1}}$ and $\frac{\beta^{\beta}v^{\beta+1}}{(\beta+1)^{\beta+1}}$ from a low-cost and high-cost seller, respectively. Given $\alpha<\beta$, expected utility is higher from a low-cost seller so a buyer prefers to solicit a bid from that seller type.

It is straightforward but tedious algebra to show that (12) (from the paper) takes the form in (24). Let us show that if $\alpha<\beta$ then the RHS of (24) exceeds the LHS. First note that the denominators are positive because $\frac{\beta}{\beta+1}>\frac{\alpha}{\alpha+1}$. Letting

$C \equiv \frac{\beta^{\alpha+1}}{(\beta+1)^{\alpha+1}}$, and $D \equiv \frac{\beta^{\beta+1}}{(\beta+1)^{\beta+1}}$, $E \equiv \frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+1}}$, and $F \equiv \frac{\alpha^{\beta+1}}{(\alpha+1)^{\beta+1}}$, (24) becomes

$$\begin{aligned}
&\frac{C-2E}{C-E} \leq \frac{D-2F}{D-F} \iff (C-2E)(D-F) \leq (C-E)(D-2F) \\
&\iff CD - 2ED - FC + 2EF \leq CD - ED - 2FC + 2EF \\
&\iff FC \leq ED \iff \frac{\alpha^{\beta+1}}{(\alpha+1)^{\beta+1}} \frac{\beta^{\alpha+1}}{(\beta+1)^{\alpha+1}} \leq \frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+1}} \frac{\beta^{\beta+1}}{(\beta+1)^{\beta+1}} \\
&\iff \frac{\alpha^{\beta-\alpha}}{(\alpha+1)^{\beta-\alpha}} \leq \frac{\beta^{\beta-\alpha}}{(\beta+1)^{\beta-\alpha}} \iff \left(\frac{\alpha}{\alpha+1}\right)^{\beta-\alpha} \leq \left(\frac{\beta}{\beta+1}\right)^{\beta-\alpha}
\end{aligned}$$

which holds because $\beta > \alpha$ and $\frac{\beta}{\beta+1} \geq \frac{\alpha}{\alpha+1}$.

Next we prove that if $\alpha < 1$ then $E\left[\pi^{\text{coll}}\right] > E\left[\pi^{\text{comp}}\right]$. Under competition, the ex-ante expected profit for a seller is

$$\begin{aligned}
E\left[\pi^{\text{comp}}\right] =& q\left(1-\frac{q}{2}\right) \int_0^1 \int_0^{\frac{\alpha}{\alpha+1}v} \left(\frac{\alpha}{\alpha+1}v - c\right) \alpha c^{\alpha-1} dc dv \\
&+ (1-q)\left(\frac{1-q}{2}\right) \int_0^1 \int_0^{\frac{\beta}{\beta+1}v} \left(\frac{\beta}{\beta+1}v - c\right) \beta c^{\beta-1} dc dv
\end{aligned}$$

$$E\left[\pi^{\text{comp}}\right] = q\left(1-\frac{q}{2}\right)\left(\frac{\alpha^{\alpha+1}}{(\alpha+2)(\alpha+1)^{\alpha+2}}\right) + (1-q)\left(\frac{1-q}{2}\right)\left(\frac{\beta^{\beta+1}}{(\beta+2)(\beta+1)^{\beta+2}}\right). \tag{5}$$

The expected collusive profit is

$$E\left[\pi^{\text{coll}}\right] = \frac{1}{2}\int_0^1 \left[q \int_0^{\frac{\beta}{\beta+1}v} \left(\frac{\beta}{\beta+1}v - c\right) \alpha c^{\alpha-1} dc + (1-q) \int_0^{\frac{\beta}{\beta+1}v} \left(\frac{\beta}{\beta+1}v - c\right) \beta c^{\beta-1} dc\right] dv$$

$$E\left[\pi^{\text{coll}}\right] = \frac{1}{2} q \frac{1}{(\alpha+1)(\alpha+2)} \left(\frac{\beta}{\beta+1}\right)^{\alpha+1} + \frac{1}{2}(1-q) \frac{1}{(\beta+1)(\beta+2)} \left(\frac{\beta}{\beta+1}\right)^{\beta+1}. \tag{6}$$

We require (6) > (5) which can be shown to be equivalent to

$$\begin{aligned}
& q\left(\frac{\beta^{\beta+1}}{(\beta+2)(\beta+1)^{\beta+2}} - \frac{\alpha^{\alpha+1}}{(\alpha+2)(\alpha+1)^{\alpha+2}}\right) \qquad (7) \\
&\quad < \frac{\beta^{\beta+1}}{(\beta+2)(\beta+1)^{\beta+2}} - 2\frac{\alpha^{\alpha+1}}{(\alpha+2)(\alpha+1)^{\alpha+2}} + \frac{1}{(\alpha+1)(\alpha+2)} \left(\frac{\beta}{\beta+1}\right)^{\alpha+1}
\end{aligned}$$

If $\frac{\beta^{\beta+1}}{(\beta+2)(\beta+1)^{\beta+2}} - \frac{\alpha^{\alpha+1}}{(\alpha+2)(\alpha+1)^{\alpha+2}} > 0$ then (7) becomes

$$q < 1 + \frac{\frac{1}{(\alpha+1)(\alpha+2)} \left(\frac{\beta}{\beta+1}\right)^{\alpha+1} - \frac{\alpha^{\alpha+1}}{(\alpha+2)(\alpha+1)^{\alpha+2}}}{\frac{\beta^{\beta+1}}{(\beta+2)(\beta+1)^{\beta+2}} - \frac{\alpha^{\alpha+1}}{(\alpha+2)(\alpha+1)^{\alpha+2}}}.$$

As the RHS is always greater than 1, this condition holds. If instead $\frac{\beta^{\beta+1}}{(\beta+2)(\beta+1)^{\beta+2}} - \frac{\alpha^{\alpha+1}}{(\alpha+2)(\alpha+1)^{\alpha+2}} < 0$ then (7) becomes

$$q > \frac{\frac{\beta^{\beta+1}}{(\beta+2)(\beta+1)^{\beta+2}} - 2\frac{\alpha^{\alpha+1}}{(\alpha+2)(\alpha+1)^{\alpha+2}} + \frac{1}{(\alpha+1)(\alpha+2)}\left(\frac{\beta}{\beta+1}\right)^{\alpha+1}}{\frac{\beta^{\beta+1}}{(\beta+2)(\beta+1)^{\beta+2}} - \frac{\alpha^{\alpha+1}}{(\alpha+2)(\alpha+1)^{\alpha+2}}}. \tag{8}$$

By assumption the denominator is negative. Letting $\Lambda(\beta)$ denote the numerator, note that $\Lambda(\alpha) = 0$ and it can be shown that $\Lambda'(\beta) > 0$ when $\alpha < 1$. Hence, $\Lambda(\beta) \geq \Lambda(\alpha) = 0$ which implies the RHS of (8) is negative in which case it is always true. Thus, a sufficient condition for collusion to be profitable is that the low-cost distribution is concave: $\alpha < 1$.